Answers of Homework of Lecture 6-7 for Reference

Theorem 6.8 (Properties of $\Phi(t,t_0)$) (Homework)

1) $\Phi(t,t_0)=\Phi(t)\Phi^{-1}(t_0)=\Psi(t)\Psi^{-1}(t_0)$;

2) $\Phi(t,t_0)=\Phi(t,t_1)\Phi(t_1,t_0)$;

3) $\Phi^{-1}(t,t_0)=\Phi(t_0,t)$;

4) $x(t,t_0,x_0)=\Phi(t,t_0)x_0$

Proof. 1) Since $\Phi(t,t_0)$ and $\Phi(t)$ are both fundamental matrix solutions of $x'=A(t)x$, there exists a nonsingular matrix C such that

$$\Phi(t,t_0)=\Phi(t)C.$$

Moreover, $\Phi(t_0,t_0)=I$. Then $I=\Phi(t_0)C$, i.e. $C=\Phi^{-1}(t_0)$. Therefore,

$$\Phi(t,t_0)=\Phi(t)\Phi^{-1}(t_0).$$

It is the same to show that $\Phi(t,t_0)=\Psi(t)\Psi^{-1}(t_0)$.

2) Based on 1), $\forall\, t,\ t_0,\ t_1\in I$, $\Phi(t,t_1)=\Phi(t)\Phi^{-1}(t_0)\Phi(t_0)\Phi^{-1}(t_1)=\Phi(t,t_0)\Phi(t_0,t_1)$.

3) $\forall\, t,t_0\in I$, $\Phi^{-1}(t_0,t)=\{\Phi(t_0)\Phi^{-1}(t)\}^{-1}=\Phi(t)\Phi^{-1}(t_0)=\Phi(t,t_0)$.

4) Since the solutions $x(t,t_0,x_0)$ and $\Phi(t,t_0)x_0$ satisfy

$$x(t_0,t_0,x_0)=x_0=\Phi(t_0,t_0)x_0,$$

it implies that $x(t,t_0,x_0)=\Phi(t,t_0)x_0$ by uniqueness. □

1. Show that $\dot{x}=A(t)x+h(t)$ has only $n+1$ linearly independent solutions, where $h(t)$ is not identically zero on I; $A(t)$ and $h(t)$ are continuous on I.

Proof. Since $A(t)$ and $h(t)$ are continuous on I, there exists a basis $\{x_j(t),t\in I\}\in\Omega$, $j=1,2,\cdots,n$ for $\dot{x}=A(t)x$. Suppose that $x(t)$ is a particular solution of $\dot{x} = A(t)x + h(t)$, which is guarantied by the variation of constant, i.e.

$$x(t) = \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s)h(s)\,\mathrm{d}s .$$

Then $x_j(t)+x(t)$ ($j=1,2,\cdots,n$) are n solutions of $\dot{x}=A(t)x+h(t)$ by Superposition Principle. Therefore, we have obtained $n+1$ solutions of $\dot{x} = A(t)x + h(t)$ given by

$$x_j(t)+x(t) \quad (j=1,2,\cdots,n) \text{ and } x(t), \quad t\in I .$$

We are going to show them linearly independent on I. If

$$\sum_{j=0}^{n} c_j(x_j(t)+x(t)+c_{n+1}x(t)\equiv 0 \quad \text{for} \quad t\in I$$

$$\Leftrightarrow \quad \{\sum_{j=0}^{n+1} c_j\}x(t)\equiv -\sum_{j=1}^{n} c_j x_j(t) .$$

If $\sum_{j=0}^{n+1} c_j \neq 0$, we have $x(t)\equiv -\{\sum_{j=0}^{n+1} c_j\}^{-1}\sum_{j=1}^{n} c_j x_j(t)$. This yields that $x(t)\in\Omega$ by Superposition Principle, which is not possible unless $h(t)\equiv 0$. This is a contradiction. This contradiction implies $\sum_{j=0}^{n+1} c_j = 0$. Then, $\sum_{j=1}^{n} c_j x_j(t)\equiv 0$ for $t\in I$. Since $\{x_j(t)\}$ is a basis of Ω by assumption, it yields $c_1 = c_2 = \cdots = c_n = 0$. Then we have $c_{n+1} = 0$ by $\sum_{j=0}^{n+1} c_j = 0$. It therefore concludes that $\{x_j(t)+x(t)\}$ and $x(t)$ are linearly independent. The existence is shown.

Next we show the "only". We show it by contradiction. If there exist $n+2$ linearly independent solutions $\varphi_0(t)$, $\varphi_1(t),\cdots,$ $\varphi_{n+1}(t)$ of $\dot{x} = A(t)x + h(t)$ for $t\in I$. Then

$$x_1(t)=\varphi_1(t)-\varphi_0(t), \quad x_2(t)=\varphi_2(t)-\varphi_0(t),\cdots, x_{n+1}(t)=\varphi_{n+1}(t)-\varphi_0(t)$$

are $n+1$ solutions of $\dot{x} = A(t)x$ for $t\in I$ by Superposition Principle. Since $\dot{x} = A(t)x$ has only n dimension, then any $n+1$ solutions, including $x_1(t)$, $x_2(t),\cdots, x_{n+1}(t)$ must be linear dependent on $t\in I$ by the fundamental theorem. Then, there exist c_j ($j=1,\cdots,n+1$), not all zero, such that

$$\sum_{j=1}^{n+1} c_j x_j(t) \equiv 0, \quad t \in I.$$

That is,

$$\sum_{j=1}^{n+1} c_j \varphi_j(t) - \{\sum_{j=1}^{n+1} c_j\}\varphi_0(t) \equiv 0, \quad t \in I.$$

Denote $c_0 = -\sum_{j=1}^{n+1} c_j$. The above equation is now $\sum_{j=0}^{n+1} c_j \varphi_j(t) \equiv 0$, $t \in I$. In which, $\{c_j\}$ ($j = 0, 1, \cdots, n+1$) are not all zero. It shows by definition that $\varphi_0(t)$, $\varphi_1(t), \cdots, \varphi_{n+1}(t)$ are linearly dependent on $t \in I$. This is a contradiction to the assumption. Therefore, $\dot{x} = A(t)x + h(t)$ has only $n+1$ linearly independent solutions on $t \in I$. The proof is finished. □

2. Show that the IVP

$$\dot{x} = A(t)x + f(t, x), \quad x(t_0) = x_0$$

and the integral equations

$$x(t) = \Phi(t)\Phi^{-1}(t_0)x_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s) f(s, x(s))\,\mathrm{d}s$$

are equivalent. That is, they have the same set of solutions, where $\Phi(t)$ is a fundamental matrix solution of $\dot{x} = A(t)x$, where $A(t)$ is continuous on I and $f(t, x)$ is continuous on $I \times R^n$.

Proof. Suppose that $x = \varphi(t)$ is a continuous solution of the integral equations, then, $\varphi(t_0) = x_0$ and

$$\varphi(t) \equiv \Phi(t)\Phi^{-1}(t_0)x_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s) f(s, \varphi(s))\,\mathrm{d}s.$$

Since $\varphi(t)$ is continuous and $\Phi^{-1}(t) f(t, \varphi(t))$ is continuous, we conclude that $\varphi(t)$ is differentiable. Taking derivative of t on both side of the integral equations yields

$$\varphi'(t) \equiv \Phi'(t)\Phi^{-1}(t_0)x_0 + \int_{t_0}^{t} \Phi'(t)\Phi^{-1}(s) f(s, \varphi(s))\,\mathrm{d}s + \Phi(t)\Phi^{-1}(t) f(t, \varphi(t))$$

$$= A(t)\Phi(t)\Phi^{-1}(t_0)x_0 + A(t)\int_{t_0}^{t} \Phi(t)\Phi^{-1}(s) f(s,\varphi(s))\,\mathrm{d}s + f(t,\varphi(t))$$

$$= A(t)\{\Phi(t)\Phi^{-1}(t_0)x_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s) f(s,\varphi(s))\,\mathrm{d}s\} + f(t,\varphi(t))$$

$$= A(t)\varphi(t) + f(t,\varphi(t)).$$

Therefore, $x = \varphi(t)$ is a solution of $\dot{x} = A(t)x + f(t,x)$ with $\varphi(t_0) = x_0$.

Conversely, suppose that $x = \varphi(t)$ is a solution of $\dot{x} = A(t)x + f(t,x)$ with $\varphi(t_0) = x_0$. It needs to show

$$\varphi(t) \equiv \Phi(t)\Phi^{-1}(t_0)x_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s) f(s,\varphi(s))\,\mathrm{d}s,$$

$$\Leftrightarrow \quad \Phi^{-1}(t)\varphi(t) \equiv \Phi^{-1}(t_0)x_0 + \int_{t_0}^{t} \Phi^{-1}(s) f(s,\varphi(s))\,\mathrm{d}s;$$

$$\Leftrightarrow \quad \Phi^{-1}(t)\varphi(t) - \Phi^{-1}(t_0)x_0 \equiv \int_{t_0}^{t} \Phi^{-1}(s) f(s,\varphi(s))\,\mathrm{d}s;$$

$$\Leftrightarrow \quad \Phi^{-1}(t)\varphi(t)\,|_{t_0}^{t} \equiv \int_{t_0}^{t} \Phi^{-1}(s) f(s,\varphi(s))\,\mathrm{d}s.$$

The last equation can be obtained by integrating $\{\Phi^{-1}(t)\varphi(t)\}' \equiv \Phi^{-1}(t) f(t,\varphi(t))$ from t_0 to t.

Since

$$\{\Phi^{-1}(t)\varphi(t)\}' = \Phi^{-1}(t)\varphi'(t) + \{\Phi^{-1}(t)\}'\varphi(t),$$

in which we need to get the expression of $\{\Phi^{-1}(t)\}'$. To this end, it yields first

$$0 = \{\Phi(t)\Phi^{-1}(t)\}' = \Phi'(t)\Phi^{-1}(t) + \Phi(t)\{\Phi^{-1}(t)\}',$$

from the above equation, we have

$$\{\Phi^{-1}(t)\}' = -\Phi^{-1}(t)\Phi'(t)\Phi^{-1}(t) = -\Phi^{-1}(t)A(t).$$

Then,

$$\{\Phi^{-1}(t)\varphi(t)\}' = \Phi^{-1}(t)\varphi'(t) + \{\Phi^{-1}(t)\}'\varphi(t)$$

$$= \Phi^{-1}(t)\varphi'(t) - \Phi^{-1}(t)A(t)\varphi(t) = \Phi^{-1}(t)\{\varphi'(t) - A(t)\varphi(t)\}$$

$$= \Phi^{-1}(t) f(t,\varphi(t)).$$

Integrating on both sides of the above equation from t_0 to t yields

$$\{\Phi^{-1}(t)\varphi(t)\}' \equiv \Phi^{-1}(t) f(t,\varphi(t)).$$

That is,

$$\varphi(t) \equiv \Phi(t)\Phi^{-1}(t_0)x_0 + \int_{t_0}^{t} \Phi(t)\Phi^{-1}(s) f(s,\varphi(s))\,\mathrm{d}s .$$

Therefore, $x = \varphi(t)$ is a continuous solution of the integral equations. This is the end of the proof. □

3. (Lecture 7) The "Putzer Algorithm" given below is another method for computing e^{At} when we have multiple eigenvalues:

$$e^{At} = \sum_{j=0}^{n-1} r_{j+1}(t) P_j ,$$

where $P_0 = I_n$, $P_j = (A - \lambda_j I_n)(A - \lambda_{j-1} I_n)\cdots(A - \lambda_1 I_n)$, $j = 1, 2, \cdots, n$, **and** $r_j(t)$, $j = 1, 2, \cdots, n$, **are the solutions of the first-order linear differential equations and initial conditions**

$$r_1'(t) = \lambda_1 r_1(t) \quad \textbf{with} \quad r_1(0) = 1;$$

$$r_2'(t) = \lambda_2 r_2(t) + r_1(t) \quad \textbf{with} \quad r_2(0) = 0;$$

$$\cdots;$$

$$r_n'(t) = \lambda_n r_n(t) + r_{n-1}(t) \quad \textbf{with} \quad r_n(0) = 0.$$

Proof. Denote $\Phi(t) = \sum_{j=0}^{n-1} r_{j+1}(t) P_j$. Then, $\Phi(0) = \sum_{j=0}^{n-1} r_{j+1}(0) P_j = r_1(0) P_0 = I_n$ with $\det\Phi(0) = 1 \neq 0$. It remains to show that $\Phi(t)$ satisfies $\Phi'(t) = A\Phi(t)$ by uniqueness. Let $r_0(t) = 0$ for simplicity. Then,

$$\Phi'(t) = \sum_{j=0}^{n-1} r_{j+1}'(t) P_j = \sum_{j=0}^{n-1} (\lambda_{j+1} r_{j+1}(t) + r_j(t)) P_j .$$

On the other hand,

$$A\Phi(t) = \sum_{j=0}^{n-1} r_{j+1}(t) A P_j = \sum_{j=0}^{n-1} r_{j+1}(t) \{(A - \lambda_{j+1} I_n) + \lambda_{j+1} I_n\} P_j$$

$$= \sum_{j=0}^{n-1} r_{j+1}(t)(A - \lambda_{j+1} I_n) P_j + \sum_{j=0}^{n-1} \lambda_{j+1} r_{j+1}(t) P_j$$

$$= \sum_{j=0}^{n-1} r_{j+1}(t) P_{j+1} + \sum_{j=0}^{n-1} \lambda_{j+1} r_{j+1}(t) P_j$$

$$= \sum_{j=1}^{n-1} r_j(t) P_j + \sum_{j=1}^{n-1} \lambda_{j+1} r_{j+1}(t) P_j + r_n(t) P_n + \lambda_1 r_1(t) P_0$$

$$= \sum_{j=1}^{n-1} (\lambda_{j+1} r_{j+1}(t) + r_j(t)) P_j + r_n(t) P_n + (\lambda_1 r_1(t) + r_0(t)) P_0 .$$

By Hamilton-Caylay Theorem in Linear Algebra, we know that

$$P_n = (A - \lambda_n I_n)(A - \lambda_{n-1} I_n) \cdots (A - \lambda_1 I_n) = O_{n \times n} .$$

Therefore,

$$\Phi'(t) = \sum_{j=0}^{n-1} (\lambda_{j+1} r_{j+1}(t) + r_j(t)) P_j = A\Phi(t) .$$

Since $\Phi(t)$ is a fundamental matrix solution satisfying $\Phi(0) = I_n$, noting that e^{At} is also a principle matrix solution, we have by uniqueness

$$e^{At} \equiv \Phi(t) = \sum_{j=0}^{n-1} r_{j+1}(t) P_j .$$

This completes the proof. □